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Schwinger–Dyson operator of Yang–Mills matrix models with ghosts and derivations of the graded shuffle algebra

Govind S Krishnaswami[1,2,3]

[1] Institute for Theoretical Physics and Spinoza Institute, Utrecht University, Postbus 80.195, 3508 TD, Utrecht, The Netherlands
[2] Department of Mathematical Sciences and Centre for Particle Theory, Durham University, Science Site, South Road, Durham, DH1 3LE, UK
[3] Chennai Mathematical Institute, Padur PO, Siruseri 603103, India

E-mail: govind.krishnaswami@durham.ac.uk

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Abstract
We consider large-N multi-matrix models whose action closely mimics that of Yang–Mills theory, including gauge-fixing and ghost terms. We show that the factorized Schwinger–Dyson loop equations, expressed in terms of the generating series of gluon and ghost correlations $G(\xi)$, are quadratic equations $\mathcal{S}^i G = G\xi^i G$ in concatenation of correlations. The Schwinger–Dyson operator $\mathcal{S}^i$ is built from the left annihilation operator, which does not satisfy the Leibnitz rule with respect to concatenation. So the loop equations are not differential equations. We show that left annihilation is a derivation of the graded shuffle product of gluon and ghost correlations. The shuffle product is the point-wise product of Wilson loops, expressed in terms of correlations. So in the limit where concatenation is approximated by shuffle products, the loop equations become differential equations. Remarkably, the Schwinger–Dyson operator as a whole is also a derivation of the graded shuffle product. This allows us to turn the loop equations into linear equations for the shuffle reciprocal, which might serve as a starting point for an approximation scheme.

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1. Introduction

QCD, a quantum Yang–Mills theory is the best candidate for a theory of strongly interacting subatomic particles. It is an outstanding challenge to understand its non-perturbative features and their mathematical formulation. Yang–Mills theory includes as dynamical degrees of freedom, a collection of $N \times N$ matrices (gluons), where $N = 3$ in nature. The limit as the number of colours $N \to \infty$ is a promising starting point to study this theory [1]. Large-N matrix models [2] are simplified models for the dynamics of gluons.

In this paper, we establish some properties of a class of large-N multi-matrix models that may be regarded as toy-models for gauge-fixed Yang–Mills theory. These models have both Hermitian complex matrices (bosonic gluons) and Hermitian grassmann matrices (fermionic ghosts) and we call them Yang–Mills matrix models with ghosts. However, these models are generically not supersymmetric. Our work is inspired by that of Makeenko and Migdal [3–5] on the loop equations. Matrix models and their loop equations may be formulated in several different ways [6–12]. There is a large literature on matrix models involving both bosonic and fermionic degrees of freedom, especially in the context of the matrix approach to M-theory [13–15] and in the matrix regularization of the supermembrane [16].

'Solving' a matrix model can be regarded as determining the gluon and ghost correlation functions. We obtain quantum corrected equations of motion (large-N factorized Schwinger–Dyson or loop equations) for these correlations. They involve a 'classical' term linear in correlations, coming from the variation of the action. This is the 'Schwinger–Dyson operator' $\mathcal{S}^i$ acting on correlations. $\mathcal{S}^i$ is built from the left annihilation operator. The loop equations also involve a quadratic 'quantum' term in correlations, coming from the change in path integral measure. This involves concatenation products of gluon and ghost correlation tensors. However, the left annihilation operator does not satisfy the Leibnitz rule (i.e. is not a derivation) with respect to concatenation. So the loop equations are *not* differential equations. This is perhaps part of the reason why the loop equations have been difficult to solve, though they were derived for the Wilson loops of Yang–Mills theory over 25 years ago. It is therefore imperative to uncover any hidden mathematical structures of the loop equations, which may help in solving them and placing them in their natural mathematical context.

The great success of calculus in solving problems of classical mechanics is due to the fact that the equations are differential equations, rather than, say, difference equations. So it is interesting to know whether there is some limit or approximation where the loop equations become differential equations. The main result of this paper is that this is indeed the case, in the limit where concatenation of correlations is replaced by their shuffle products. While concatenation arises from concatenation of loops, shuffle arises from the point-wise product of Wilson loops. We show that this picture is a robust, in the sense that it is not spoiled by the inclusion of gauge fixing and ghost terms in the action. More precisely, we show that the left annihilation as well as the Schwinger–Dyson operator $\mathcal{S}^i$, satisfy the Leibnitz rule with respect to the (graded) shuffle product of gluon and ghost correlations. The latter allows us to reduce the nonlinear loop equations to linear equations for the shuffle-reciprocal of the generating series for correlations. Though this is not a property shared by generic matrix models, it does carry over to $3 + 1d$ gauge-fixed Yang–Mills theory. For, all we use is the algebraic structure of the action, and general properties of the large-N limit.

In a previous paper [12], we proposed an approximation scheme to compute correlation functions by solving the loop equations, in the context of bosonic matrix models. We expanded the concatenation product around the shuffle product and used the shuffle reciprocal to reduce the loop equations to linear differential equations in the shuffle algebra at the zeroth order of the expansion. In simple cases, this was shown to give a rough approximation to the exact correlations. Aside from this practical application, a mathematical lesson from our work is that it may be fruitful to view the loop equations as living in the differential bi-algebra formed by shuffle, concatenation and their derivations.

The property that $\mathcal{S}^i$ is a derivation of the shuffle product, is a finite and differential–algebraic reformulation in terms of correlation tensors of a property of large-N Yang–Mills theory mentioned in [4]. $\mathcal{S}^i$ is analogous to the path derivative of the area derivative operator, which was said to satisfy the Leibnitz rule with respect to point-wise products of Wilson loops. We hope that our alternative viewpoint and finite formulation of these infinite-dimensional notions is useful to better understand the loop equations.

2. Gluon–ghost correlations in the large-N limit

Motivated by the Lagrangian of gauge-fixed Yang–Mills theory (say in a class of covariant gauges labelled by ξ)

$$\mathcal{L} = \operatorname{tr}\left\{\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \mathrm{i}g\partial_\mu A_\nu[A^\mu, A^\nu] - \frac{g^2}{4}[A_\mu, A_\nu][A^\mu, A^\nu] \right. \\ \left. + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \partial_\mu\bar{c}\partial^\mu c - \mathrm{i}g\partial_\mu\bar{c}[A^\mu, c]\right\}, \tag{1}$$

we consider models with Λ matrices A_i, $1 \leqslant i \leqslant \Lambda$, which are $N \times N$ matrices in 'colour' space. We will call them Yang–Mills matrix models with ghosts if their action is of the form

$$\operatorname{tr} S = \frac{1}{2}\operatorname{tr} C^{ij}A_iA_j + \operatorname{tr} C^{ijk}A_i[A_j, A_k] - \frac{1}{4\alpha}\operatorname{tr}[A_i, A_j][A_k, A_l]g^{ik}g^{jl}. \tag{2}$$

We call all matrices A_i irrespective of whether they are gluons (A_μ), ghosts (c) or anti-ghosts ($\bar{c}$). Indices i, j, k are shorthand for position coordinates and polarization indices. They also specify whether the matrix is a gluon, a ghost or an anti-ghost via the ghost number of an index

$$\#(i) = \begin{Bmatrix} 0, & \text{if } A_i \text{ is a gluon;} \\ 1, & \text{if } A_i, \text{ is a ghost} \\ -1 & \text{if } A_i, \text{ is an anti-ghost} \end{Bmatrix}. \tag{3}$$

If t^α are Hermitian generators of the Lie algebra of $U(N)$, then $A_i = A_i^\alpha t^\alpha$ with $A_i^{\alpha *} = A_i^\alpha$. For a gluon, $\#(i) = 0$ and A_i^α is a real number while for a ghost or anti-ghost, $\#(i) = \pm 1$ and A_i^α is a real grassmann variable. Moreover,

$$\left(A_i^\alpha A_j^\beta\right)^* = (-1)^{\#(i)\#(j)}A_j^\beta A_i^\alpha. \tag{4}$$

Note that the ghost matrices are not related to the anti-ghost matrices by Hermiticity.

It is also useful to define the ghost number of a tensor $C^{i_1\cdots i_n}$ to be that of the multi-index: $\#(i_1\cdots i_n) = \sum_{k=1}^n \#(i_k)$.[1] In keeping with the structure of (1) we allow gluon, ghost and anti-ghost matrices in the quadratic and cubic terms of (2) but only equal numbers of ghosts and anti-ghosts in any term. In the cubic term, $[A_j, A_k]$ denotes the anti-commutator if neither A_j nor A_k is a gluon and the commutator otherwise. In the quartic term of (2) we allow only gluons. In other words, we assume g^{ij} vanishes if either i or j corresponds to a ghost or anti-ghost index and that the ghost numbers of C^{ij} and C^{ijk} vanish. (1) can be regarded as a limiting case of (2) for appropriate integral kernels C^{ij}, C^{ijk} and g^{ij}, when the indices become continuous.

The action of our matrix model will be written as $\operatorname{tr} S(A) = \operatorname{tr} S^I A_I$. It defines coupling tensors S^I. The partition function is defined as $Z = \int \Pi_j \mathrm{d}A_j \mathrm{e}^{-N\operatorname{tr} S(A)}$ where the integration is

[1] For a general tensor $G = G_I\xi^I$, $\#(G) = n$ if $\#(I) = n$ for each I for which G_I is non-vanishing. A general tensor has a well-defined ghost number only if all terms have the same ghost number. Capitals denote multi-indices $I = i_1i_2\ldots i_n$ and repeated indices are summed.

over all independent matrix elements of the gluon, ghost and anti-ghost matrices. Observables are correlation tensors

$$\left\langle \frac{\mathrm{tr}}{N} A_{I_1} \cdots \frac{\mathrm{tr}}{N} A_{I_n} \right\rangle = \frac{1}{Z} \int \Pi_j \mathrm{d}A_j \, \mathrm{e}^{-N \,\mathrm{tr}\, S(A)} \frac{\mathrm{tr}}{N} A_{I_1} \cdots \frac{\mathrm{tr}}{N} A_{I_n}. \tag{5}$$

They are symmetric in the I_k's up to a possible sign. For example,

$$\left\langle \frac{\mathrm{tr}}{N} A_I \frac{\mathrm{tr}}{N} A_J \right\rangle = (-1)^{\#(I)\#(J)} \left\langle \frac{\mathrm{tr}}{N} A_J \frac{\mathrm{tr}}{N} A_I \right\rangle. \tag{6}$$

This is because matrix elements $[A_i]^a_b$ are graded commutative. So we pick up a minus sign under transposition of ghost or anti-ghost matrices

$$[A_i]^a_b [A_j]^c_d = (-1)^{\#(i)\#(j)} [A_j]^c_d [A_i]^a_b. \tag{7}$$

It follows that the trace operation is graded cyclic, for example, $\mathrm{tr}\, A_i A_j A_k = (-1)^{\#(k)\#(ij)} \,\mathrm{tr}\, A_k A_i A_j$.

We restrict to actions $S^I A_I$ whose non-vanishing coupling tensors have zero ghost number. The gauge-fixed Yang–Mills theory (1) is of this type. In such a theory, correlations of a tensor with non-zero total ghost number must vanish[2]

$$\left\langle \frac{\mathrm{tr}}{N} A_{I_1} \cdots \frac{\mathrm{tr}}{N} A_{I_n} \right\rangle = 0 \qquad \text{if} \quad \#(I_1) + \#(I_2) + \cdots + \#(I_n) \neq 0. \tag{8}$$

For example, $\left\langle \frac{1}{N} \mathrm{tr}\, A_g A_g A_c A_c \right\rangle = 0$ but $\left\langle \frac{1}{N} \mathrm{tr}\, A_g A_c A_{\bar{c}} \right\rangle$ can be non-trivial, where g, c and $\bar{c}$ stand for gluon, ghost and anti-ghost, respectively. Multi-trace correlators factorize into single trace correlations in the large-N limit:

$$G_I = \lim_{N\to\infty} \left\langle \frac{\mathrm{tr}}{N} A_I \right\rangle, \qquad \text{and} \qquad \left\langle \frac{\mathrm{tr}}{N} A_{I_1} \cdots \frac{\mathrm{tr}}{N} A_{I_n} \right\rangle = G_{I_1} \cdots G_{I_n} + \mathcal{O}(1/N^2). \tag{9}$$

The Hermiticity (4) of the matrices in colour space implies an order reversal property under complex conjugation $G^*_{i_1 i_2 \cdots i_n} = (-1)^p G_{i_n i_{n-1} \cdots i_2 i_1}$ where $p = \#(i_1)\#(i_2 \cdots i_n) + \#(i_2)\#(i_3 \cdots i_n) + \cdots + \#(i_{n-1})\#(i_n)$. The factorized correlation tensors G_I are in general graded cyclic:

$$G_{Ii} = (-1)^{\#(i)\#(I)} G_{iI}, \qquad G_{IJ} = (-1)^{\#(I)\#(J)} G_{JI}. \tag{10}$$

Similarly, the only part of S^I that contributes is its graded cyclic projection

$$S^{Ii} \mapsto \frac{1}{|Ii|} [S^{Ii} + S^{iI} (-1)^{\#(I)\#(i)} + \cdots]. \tag{11}$$

So we assume that $S^{IJ} = S^{JI}(-1)^{\#(I)\#(J)}$ for all I, J, i.e. that S^I is graded cyclic.

3. Schwinger–Dyson to loop equations in the large-N limit

The loop equations are quantum corrected equations of motion for correlation tensors. To derive them, we consider changes of variable (vector fields)

$$A_i \mapsto A'_i = A_i + v^I_i A_I. \tag{12}$$

Here v^I_i could either be a small real number or a grassmann number with ghost number $\#(v^I_i) = \pm 1$. However, we cannot change a complex number by a grassmann-valued quantity and vice versa, so we require that v^I_i be such that

$$\#(i) = \#(v^I_i) + \#(I). \tag{13}$$

[2] Consequence of Feynman rules: interaction vertices have equal numbers of ghost and anti-ghost lines attached.

An example of such a change of variable is a BRST transformation in Yang–Mills theory

$$A_\mu^\alpha \mapsto A_\mu^\alpha - \frac{1}{g}(D_\mu c^\alpha)\lambda;\ c^\alpha \mapsto c^\alpha - \frac{1}{2} f^{\alpha\beta\gamma} c^\beta c^\gamma \lambda;\ \bar{c}^\alpha \mapsto \bar{c}^\alpha - \frac{1}{\xi g}\left(\partial^\mu A_\mu^\alpha\right)\lambda. \tag{14}$$

Here $v_i^I \propto \lambda$ is a constant grassmann quantity with ghost number -1, i.e. $\#\left(v_i^I\right) = -1$ and it is easily seen that the conditions $\#(i) = \#\left(v_i^I\right) + \#(I)$ are satisfied for the BRST transformation.

Under (12), $A_K \mapsto A_K + \delta_K^{LiM} v_i^I A_{LIM}$. So if we define the 'loop' variable $\Phi_K = \frac{1}{N}\operatorname{tr} A_K$,

$$\begin{aligned}
&\Phi_K \mapsto \Phi_K + \delta_K^{LiM} v_i^I \Phi_{LIM} \\
&\Phi_{K_1}\cdots\Phi_{K_n} \mapsto \Phi_{K_1}\cdots\Phi_{K_n} + \sum_{p=1}^{n} \delta_{K_p}^{L_p i M_p} v_i^I \Phi_{L_p I M_p} \\
&\mathrm{e}^{-N\operatorname{tr} S^J A_J} \mapsto \mathrm{e}^{-N\operatorname{tr} S^J A_J}\left[1 - N^2 v_i^I S^{J_1 i J_2}\Phi_{J_1 I J_2} + \cdots\right] \\
&\det\left(\frac{\partial A_i'}{\partial A_j}\right) = \det\left(\frac{\partial [A_i]_b^a + v_i^I [A_I]_b^a}{\partial [A_j]_d^c}\right) \\
&\qquad = \det\left(\delta_i^j \delta_c^a \delta_b^d + v_i^I (-1)^{\#(j)\#(I_1)} \delta_I^{I_1 j I_2} [A_{I_1}]_c^a [A_{I_2}]_b^d\right) \\
&\qquad = 1 + N^2 v_i^I \delta_I^{I_1 i I_2} (-1)^{\#(i)\#(I_1)} \Phi_{I_1}\Phi_{I_2} + \cdots.
\end{aligned} \tag{15}$$

The sign in the Jacobian comes from moving the grassmann (left) derivative through the product. Requiring the invariance of (5) under the changes of integration variables (12) leads to the finite-*N* Schwinger–Dyson equations (SDE)

$$v_i^I S^{J_1 i J_2}\left\langle \Phi_{J_1 I J_2}\right\rangle = v_i^{I_1 i I_2}(-1)^{\#(i)\#(I_1)}\left\langle \Phi_{I_1}\Phi_{I_2}\right\rangle + \frac{1}{N^2}\sum_{p=1}^{n} \delta_{K_p}^{L_p i M_p} v_i^I \left\langle \Phi_{L_p I M_p}\right\rangle. \tag{16}$$

In the large-*N* limit, the factorized SDE ignoring the last term are

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = v_i^{I_1 i I_2}(-1)^{\#(i)\#(I_1)} G_{I_1} G_{I_2}. \tag{17}$$

Since correlations G_{I_1} vanish if $\#(I_1) \neq 0$ we get

$$v_i^I S^{J_1 i J_2} G_{J_1 I J_2} = v_i^{I_1 i I_2} G_{I_1} G_{I_2} \quad \text{for all vector fields } v. \tag{18}$$

Now taking v_i^I to be non-vanishing only for a fixed i, I gives us the loop equations (LE)

$$S^{J_1 i J_2} G_{J_1 I J_2} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2} \quad \forall I, i. \tag{19}$$

Using graded cyclicity of S^I and G_I we write this as ($|I|$ is the length of the multi-index *I*)

$$|Ji| S^{Ji} G_{JI} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}. \tag{20}$$

The loop equations have the same form as for a bosonic matrix model (see [12]).

4. Loop equations in terms of concatenation and left annihilation

Define the left annihilation operator acting on the generating series of correlations $G(\xi) = G_I \xi^I$

$$[D_j G]_I = G_{jI} \qquad \text{and} \qquad \left[D_{j_n} D_{j_{n-1}} \cdots D_{j_1} G\right]_I = \left[D_{j_n \cdots j_1} G\right]_I = G_{j_1 \cdots j_n I}. \tag{21}$$

The ghost number $\#(D_j G) = \#(j)$ if $G(\xi)$ has zero ghost number. In terms of the concatenation product

$$[FG]_I = \delta_I^{JK} F_J G_K; \qquad F(\xi)G(\xi) = F_I G_J \xi^{IJ}, \tag{22}$$

the LE take the same form as in the purely bosonic case [12]:

$$\sum_{n\geqslant 0}(n+1)S^{j_1\cdots j_n i}D_{j_n\cdots j_1}G(\xi) = G(\xi)\xi^i G(\xi), \qquad \text{or} \qquad \mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi)$$

$$\text{where} \quad \mathcal{S}^i = \sum_{n\geqslant 0}(n+1)S^{j_1\cdots j_n i}D_{j_n\cdots j_1}. \tag{23}$$

Both LHS and RHS have ghost number $\#(i)$, provided $G(\xi)$ and S^I have zero ghost number.

Graded commutator To simplify the notation below, we introduce the graded commutator

$$[D_i, D_j] = D_i D_j - (-1)^{\#(i)\#(j)} D_j D_i. \tag{24}$$

It is graded symmetric $[D_i, D_j] = -(-1)^{\#(i)\#(j)}[D_j, D_i]$, and reduces to the commutator if either i or j is a gluon and to the anti-commutator if neither is a gluon. We use the same notation for commutators and anti-commutators of the A_i.

5. Example: ghost terms in Yang–Mills action

As an example, consider a class of matrix models inspired by the ghost terms in the gauge-fixed Yang–Mills action (1) $\partial_\mu \bar{c}\partial^\mu c - ig\partial_\mu\bar{c}[A^\mu, c]$. The action is

$$\operatorname{tr} S = \tfrac{1}{2}\operatorname{tr} C^{ij}A_iA_j + \operatorname{tr} C^{ijk}A_i[A_j, A_k] \tag{25}$$

where [,] is the graded commutator (24). If all the matrices are gluons, then this is a zero-momentum Gaussian + Chern–Simons-type matrix model. Here we allow gluon, ghost and anti-ghost matrices so that for special choices of C^{ij} and C^{ijk}, this action can also model the terms $\partial_\mu\bar{c}\partial^\mu c$ and $\mathrm{i}g\partial_\mu\bar{c}[A^\mu, c]$ appearing in the gauge-fixed Yang–Mills action. We obtain the coupling tensors S^{ij} and S^{ijk}, which can be chosen to be graded-cyclic. Then we obtain the factorized LE and the differential operator $\mathcal{S}^i$. First, write the action as

$$\begin{aligned}\operatorname{tr} S &= \tfrac{1}{2}\operatorname{tr} C^{ij}A_iA_j + \operatorname{tr} C^{ijk}A_i\{A_jA_k - (-1)^{\#(j)\#(k)}A_kA_j\} \\ &= \tfrac{1}{2}\operatorname{tr} C^{ij}A_iA_j + \operatorname{tr}[C^{ijk}A_{ijk} - (-1)^{\#(j)\#(k)}C^{ikj}A_{ijk}].\end{aligned} \tag{26}$$

Thus the coupling tensors are

$$S^{ij} = \tfrac{1}{2}C^{ij} \qquad \text{and} \qquad S^{ijk} = C^{ijk} - (-1)^{\#(j)\#(k)}C^{ikj}. \tag{27}$$

From the graded cyclicity of the trace, it follows that the coupling tensors are graded cyclic

$$S^{ij} = (-1)^{\#(i)\#(j)}S^{ji} \qquad \text{and} \qquad S^{ijk} = (-1)^{\#(k)\#(ij)}S^{kij}. \tag{28}$$

As for the differential operator $\mathcal{S}^i$, from (23) we have

$$\mathcal{S}^i = C^{ji}D_j + 3S^{jki}D_{kj} = C^{ji}D_j + 3(C^{jki} - (-1)^{\#(k)\#(i)}C^{jik})D_kD_j. \tag{29}$$

The interesting question (which we will answer in the affirmative in section 8) is whether $\mathcal{S}^i$ is a derivation of the graded shuffle product of ghost number zero tensors. Mere graded cyclicity of S^{ijk} is not sufficient for this. Rather, it is useful to write $\mathcal{S}^i$ in terms of graded commutators of left annihilations. The reason is that the left annihilation D_i is like a first-order differential operator (vector field) when acting on the shuffle algebra. While products of vector fields D_iD_j are no longer vector fields, their commutators $[D_i, D_j]$ continue to be vector fields and therefore define derivations of the shuffle algebra.

In terms of the graded commutator (24), the action may be written as

$$\operatorname{tr} S = \operatorname{tr}\tfrac{1}{2}C^{ij}A_iA_j + \operatorname{tr} C^{ijk}A_i[A_j, A_k] = \operatorname{tr}\tfrac{1}{2}C^{ij}A_iA_j + \operatorname{tr} C^{ijk}[A_i, A_j]A_k. \tag{30}$$

Graded symmetry of the graded commutator in turn implies graded symmetry of C^{ijk}

$$C^{ijk} + (-1)^{\#(j)\#(k)} C^{ikj} = 0 \qquad \text{and} \qquad C^{ijk} + (-1)^{\#(i)\#(j)} C^{jik} = 0. \tag{31}$$

More precisely, even if these quantities were non-vanishing, they would not contribute to the action. Therefore they can be taken to vanish without loss of generality. Using this property twice we can write

$$\begin{aligned}\mathcal{S}^i &= C^{ji} D_j + 3C^{jki} D_k D_j - 3(-1)^{\#(j)\#(i)} C^{kij} D_j D_k \\ &= C^{ji} D_j + 3C^{jki} D_k D_j - 3(-1)^{\#(k)\#(j)} C^{jki} D_j D_k = C^{ji} D_j + 3C^{ijk}[D_k, D_j].\end{aligned} \tag{32}$$

Thus $\mathcal{S}^i$ is a linear combination of the left annihilation and its (anti-)commutators. Finally, the factorized LE for the action (25) are

$$C^{ji} D_j G(\xi) + 3C^{ijk}[D_k, D_j]G(\xi) = G(\xi)\xi^i G(\xi). \tag{33}$$

More generally, $\mathcal{S}^i$ for the full gauge-fixed Yang–Mills action is a linear sum of the following combinations of left annihilation operators

$$D_g, D_c, D_{\bar{c}}, [D_c, D_{\bar{c}}]_+, [D_c, D_g]_-, [D_{\bar{c}}, D_g]_-, [D_{g_1}, D_{g_2}]_-, [D_{g_1}, [D_{g_2}, D_{g_3}]_-]_-. \tag{34}$$

Each of these combinations arises as the variation of one or more terms in the gauge-fixed Yang–Mills action (1). D_g comes from varying terms involving two derivatives and two gluon fields (e.g. $\frac{1}{2\xi}(\partial^\mu A_\mu)^2$ and $\partial_\mu A_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu)$). D_c and $D_{\bar{c}}$ come from $\partial_\mu \bar{c}\partial^\mu c$. The anti-commutator $[D_c, D_{\bar{c}}]_+$ arises from the variation of $g\partial_\mu \bar{c}[A^\mu, c]$ with respect to the gluon. The commutators $[D_c, D_g]$ and $[D_{\bar{c}}, D_g]$ arise from varying the same term with respect to an anti-ghost or a ghost (as in the example above). $[D_{g_1}, D_{g_2}]$ originates from varying the term linear in derivatives $g\partial_\mu A_\nu[A^\mu, A^\nu]$. Finally $[D_{g_1}, [D_{g_2}, D_{g_3}]]$ has its origin in the term independent of momentum $\frac{g^2}{4}[A_\mu, A_\nu][A^\mu, A^\nu]$.

We showed in [12] that $\mathcal{S}^i$ for the purely gluonic part of the Yang–Mills + Chern–Simons + Gaussian action is a derivation of the shuffle product of gluon correlations. Here we show that even when gauge-fixing and ghost terms are included, $\mathcal{S}^i$ is a derivation of the graded shuffle product of gluon–ghost correlations.

6. Graded shuffle product

We will call the extension of the shuffle product (see [12, 17]) to correlation tensors of gluon and ghost matrices by the name graded shuffle product. It is essentially the shuffle product with minus signs when ghost or anti-ghost indices are transposed. The definition is

$$[F \circ G]_I \equiv \sum_{I_1 \sqcup I_2 = I} (-1)^{\gamma(I; I_1, I_2)} F_{I_1} G_{I_2}. \tag{35}$$

$I_1 \sqcup I_2 = I$ is the condition that I_1 and I_2 are complementary order preserving sub-strings of I. In other words, we riffle-shuffle the card packs I_1 and I_2. For example, if $I = i_1 i_2 i_3 i_4$ then one permissible choice is $I_1 = i_2 i_4$ and $I_2 = i_1 i_3$ while $I_1 = i_3 i_2$, $I_2 = i_1 i_4$ is not allowed. We call $\gamma(I; I_1, I_2)$ the *ghost crossing number* of the ordered triple $(I; I_1, I_2)$. It is just zero for bosonic matrix models. More generally, the string I is transformed into the string $I_1 I_2$ by a minimum number of transpositions of neighbouring indices. Each transposition $i_p i_q \mapsto i_q i_p$ contributes $\#(i_p)\#(i_q)$. $\#(i_p)\#(i_q)$ is 0 if both i_p, i_q are gluons, +1 if both are ghosts or anti-ghosts and -1 if one was a ghost and the other an anti-ghost. The sum of these contributions is the ghost crossing number. For example, let

$$I = i_1 i_2 i_3 i_4 i_5; \quad I_1 = i_1 i_4 i_5; \quad I_2 = i_2 i_3. \tag{36}$$

The sequence of transpositions may be

$$\begin{aligned} i_1i_2i_3i_4i_5 &\to i_1i_2i_4i_3i_5(-1)^{\#(i_4)\#(i_3)} \to i_1i_4i_2i_3i_5(-1)^{\#(i_4)\#(i_3)+\#(i_4)\#(i_2)} \\ &\to i_1i_4i_2i_5i_3(-1)^{\#(i_4)\#(i_3)+\#(i_4)\#(i_2)+\#(i_5)\#(i_3)} \\ &\to i_1i_4i_5i_2i_3(-1)^{\#(i_4)\#(i_3)+\#(i_4)\#(i_2)+\#(i_5)\#(i_3)+\#(i_5)\#(i_2)} \\ &\to i_1i_4i_5i_2i_3(-1)^{\#(i_4i_5)\#(i_2i_3)}. \end{aligned} \tag{37}$$

The *ghost crossing sign* $(-1)^{\gamma(I;I_1,I_2)}$ is independent of the choice of sequence of transpositions, so we do not need to stick to the minimum number to find the sign. Moreover,

$$(-1)^{\gamma(I;J,K)} = (-1)^{\gamma(I;K,J)+\#(J)\#(K)}. \tag{38}$$

In the following we will be interested in tensors G_I of zero ghost number, since the others vanish.

Preservation of zero ghost number. The graded shuffle product of two tensors of zero ghost number is again a tensor of zero ghost number. Suppose F_{I_1} and G_{I_2} have zero ghost numbers, $\#(I_1) = \#(I_2) = 0$. From (35) and (3), if $I = I_1 \sqcup I_2$, then $\#(I) = \#(I_1) + \#(I_2) = 0$. So $(F \circ G)_I$ has zero ghost number.

Commutativity of graded shuffle. The graded shuffle product $F \circ G$ is commutative if either F or G has zero ghost number:

$$\begin{aligned} [F \circ G]_I &= \sum_{I_1 \sqcup I_2 = I} (-1)^{\gamma(I;I_1,I_2)} F_{I_1} G_{I_2} = \sum_{I_1 \sqcup I_2 = I} (-1)^{\gamma(I;I_2,I_1)+\#(I_1)\#(I_2)} F_{I_1} G_{I_2} \\ &= \sum_{I_2 \sqcup I_1 = I} (-1)^{\gamma(I;I_1,I_2)+\#(I_2)\#(I_1)} F_{I_2} G_{I_1} = \sum_{I_1 \sqcup I_2 = I} (-1)^{\gamma(I;I_1,I_2)} G_{I_1} F_{I_2} = [G \circ F]_I. \end{aligned} \tag{39}$$

7. Left annihilation is a derivation of graded shuffle product

D_i is a derivation of the graded shuffle product of two ghost number zero tensors, $D_i(F \circ G) = (D_i F) \circ G + F \circ (D_i G)$ if $\#(F) = \#(G) = 0$. To show this, we write

$$[D_i(F \circ G)]_I = [F \circ G]_{iI} = \sum_{I_1 \sqcup I_2 = iI} (-1)^{\gamma(iI;I_1,I_2)} F_{I_1} G_{I_2}. \tag{40}$$

Now either $i \in I_1$ or $i \in I_2$, so

$$\begin{aligned} [D_i(F \circ G)]_I &= \sum_{I_1 \sqcup I_2 = I} \left[(-1)^{\gamma(iI;iI_1,I_2)} F_{iI_1} G_{I_2} + (-1)^{\gamma(iI;I_1,iI_2)} F_{I_1} G_{iI_2}\right] \\ &= \sum_{I_1 \sqcup I_2 = I} \left[(-1)^{\gamma(iI;iI_1,I_2)} [D_i F]_{I_1} G_{I_2} + (-1)^{\gamma(iI;I_1,iI_2)} F_{I_1} [D_i G]_{I_2}\right]. \end{aligned} \tag{41}$$

Since i does not cross any index, $(-1)^{\gamma(iI;iI_1,I_2)} = (-1)^{\gamma(I;I_1,I_2)}$. Similarly, $(-1)^{\gamma(iI;I_1,iI_2)} = (-1)^{\#(i)\#(I_1)+\gamma(I;I_1,I_2)}$. Thus

$$[D_i(F \circ G)]_I = \sum_{I_1 \sqcup I_2 = I} \left[(-1)^{\gamma(I;I_1,I_2)} [D_i F]_{I_1} G_{I_2} + (-1)^{\#(i)\#(I_1)+\gamma(I;I_1,I_2)} F_{I_1} [D_i G]_{I_2}\right]. \tag{42}$$

Since correlation tensors vanish for non-zero ghost number we can take $\#(I_1) = 0$. Hence $[D_i(F \circ G)]_I = [(D_i F) \circ G]_I + [F \circ (D_i G)]_I$. Thus D_i is a derivation of the shuffle product of two tensors provided each has zero ghost number. More generally, if no assumption is made on the ghost number of F and G, then

$$[D_i(F \circ G)]_I = [(D_i F) \circ G]_I + \sum_{I_1 \sqcup I_2 = I} (-1)^{\#(i)\#(I_1)+\gamma(I;I_1,I_2)} F_{I_1} [D_i G]_{I_2}. \tag{43}$$

Graded commutator of left annihilation. In the bosonic theory, a commutator $[D_i, D_j]$ of derivations is a derivation of the shuffle product [12]. More generally, we will show that if F, G have zero ghost number, then the graded commutator of left annihilations is a derivation of their shuffle product:

$$[D_i, D_j](F \circ G) = ([D_i, D_j]F) \circ G + F \circ ([D_i, D_j]G). \tag{44}$$

To show this, we use the derivation property of D_j and then the more general result (43) for D_i.

$$\begin{aligned} &D_i D_j(F \circ G) = D_i(D_j F \circ G) + D_i(F \circ D_j G) \\ &[D_i D_j(F \circ G)]_I = [D_{ij} F \circ G]_I + [D_i F \circ D_j G]_I + \sum_{I_1 \sqcup I_2 = I} (-1)^{\#(i)\#(I_1)+\gamma(I;I_1,I_2)} \\ &\qquad \times [D_j F]_{I_1} [D_i G]_{I_2} + \sum_{I_1 \sqcup I_2 = I} (-1)^{\#(i)\#(I_1)+\gamma(I;I_1,I_2)} F_{I_1} [D_{ij} G]_{I_2}. \end{aligned} \tag{45}$$

In the third term $\#(I_1) = -\#(j)$ and in the last term, $\#(I_1) = 0$, thus

$$\begin{aligned} D_{ij}(F \circ G) &= D_{ij} F \circ G + F \circ D_{ij} G + D_i F \circ D_j G + (-1)^{\#(i)\#(j)} D_j F \circ D_i G \\ D_{ji}(F \circ G) &= D_{ji} F \circ G + F \circ D_{ji} G + D_j F \circ D_i G + (-1)^{\#(i)\#(j)} D_i F \circ D_j G. \end{aligned} \tag{46}$$

Combining these we find that the graded commutator of left annihilations is a derivation

$$\begin{aligned} [D_i, D_j](F \circ G) &= D_i D_j(F \circ G) - (-1)^{\#(i)\#(j)} D_j D_i(F \circ G) \\ &= (D_{ij} F - (-1)^{\#(i)\#(j)} D_{ji} F) \circ G + F \circ (D_{ij} G - (-1)^{\#(i)\#(j)} D_{ji} G) \\ &= ([D_i, D_j]F) \circ G + F \circ ([D_i, D_j]G). \end{aligned} \tag{47}$$

Iterated commutator of gluonic left annihilation. If i, j, k all have zero ghost numbers, then $[[D_i, D_j], D_k]$ is a derivation of the shuffle product of two tensors of zero ghost number each. This is a consequence of the derivation property of D_k and $[D_i, D_j]$ for shuffle products of tensors of zero ghost number.

$$\begin{aligned} [[D_i, D_j], D_k](F \circ G) &= [D_i, D_j](D_k F \circ G + F \circ D_k G) \\ &\quad - D_k([D_i, D_j]F \circ G) - D_k(F \circ [D_i, D_j]G). \end{aligned} \tag{48}$$

Here each of the terms within parentheses is a shuffle product of tensors of zero ghost number since i, j, k, F, G are. Applying the derivation property of D_k and $[D_i, D_j]$ again, four of the terms cancel out and we get the desired result

$$[[D_i, D_j], D_k](F \circ G) = [[D_i, D_j], D_k]F \circ G + F \circ [[D_i, D_j], D_k]G. \tag{49}$$

8. $\mathcal{S}^i$ for Yang–Mills matrix model with ghosts is a derivation of shuffle algebra

Let us now consider the Yang–Mills matrix model with ghosts introduced in (2)

$$S = \frac{1}{2} \operatorname{tr} C^{ij} A_i A_j + \operatorname{tr} C^{ijk} A_i [A_j, A_k] - \frac{1}{4\alpha} \operatorname{tr} [A_i, A_j][A_k, A_l] g^{ik} g^{jl}. \tag{50}$$

This differs from the model considered in (25) by the addition of the quartic term, which however involves only gluons. The latter was studied in [12]. Using our results from [12] and section 5 we get the loop equations $\mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi)$ where $\mathcal{S}^i$ is

$$\mathcal{S}^i = C^{ji} D_j + 3C^{ijk}[D_k, D_j] - \frac{1}{\alpha} g^{ik} g^{jl} [D_j [D_k, D_l]]. \tag{51}$$

$\mathcal{S}^i$ is a linear combination of D_j, its graded commutators and gluonic iterated commutator $[D_j[D_k, D_l]]$. Each of these was shown to be a derivation of the graded shuffle product in section 7. We conclude that the Schwinger–Dyson operator $\mathcal{S}^i$ for Yang–Mills matrix models with ghosts is a derivation of the graded shuffle product of ghost number zero correlation tensors. Based on the bosonic case in [12], one might have suspected that D_i and $\mathcal{S}^i$ would, at best, be graded derivations upon including ghosts (i.e. satisfy the Leibnitz rule up to a sign). But they turn out to be ordinary derivations of the graded shuffle product, since correlations with non-zero ghost number vanish in the large-N limit.

9. Discussion

Physically, the Schwinger–Dyson operator $\mathcal{S}^i$ arises from the variation of the action and is therefore a classical ($\hbar = 0$) concept. Indeed, the correlations in the limit $\hbar \to 0$ are annihilated by the Schwinger–Dyson operator. For Yang–Mills matrix models with ghosts, $\mathcal{S}^i = C^{ji}D_j + 3C^{ijk}[D_k, D_j] - \frac{1}{\alpha}g^{ik}g^{jl}[D_j[D_k, D_l]]$ is a linear combination of iterated (anti) commutators of the left annihilation operator. The quantum effects in the large-N limit are encoded in the variation of the measure, or the quadratic term in the loop equations $\mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi)$. ξ^i are external sources and $G(\xi)$ is the generating series of correlations, the product on the rhs is concatenation. i, j, k label the matrices and in the continuum limit would label spacetime points. We have shown that there is a mismatch between the 'classical' Schwinger–Dyson operator and the 'quantum' concatenation product, so that $\mathcal{S}^i$ is not a differential operator in the loop equations. This 'mismatch' makes the equations both interesting and hard to solve. However, we identified a new commutative shuffle product, with respect to which $\mathcal{S}^i$ is indeed a differential operator. The shuffle product is the point-wise product of Wilson loops, written in terms of correlations. As suggested in [12], we can expand concatenation around shuffle to exploit the derivation property of left annihilation D_i. At zeroth order, the loop equations become differential equations in the graded-shuffle algebra $\mathcal{S}^i G(\xi) = G(\xi) \circ \xi^i \circ G(\xi)$. Moreover, we showed that $\mathcal{S}^i$ is not just a differential operator with respect to shuffle, but behaves as a first-order differential operator. This derivation property of $\mathcal{S}^i$ leads to a further simplification. First we define the reciprocal of $G(\xi)$ with respect to the graded shuffle product. Since $G_0 = 1$ is non-vanishing, $G(\xi)$ has a right reciprocal $G(\xi) \circ F(\xi) = 1$. Moreover, non-vanishing G_I have zero ghost number, so the same is true of the F_I. Thus, the graded shuffle product $G \circ F$ is commutative and the reciprocal is unique and two-sided. Explicitly, $F_0 = 1$, $F_i = -G_i$, $F_{ij} = -G_{ij} + G_i G_j\{1 + (-1)^{\#(i)\#(j)}\}$ etc. More generally, for $|I| \geqslant 1$,

$$F_I = -G_I - \sum_{\substack{I_1 \sqcup I_2 = I \\ I_1 \neq I, I_2 \neq I}} (-1)^{\gamma(I, I_1, I_2)} F_{I_1} G_{I_2} \tag{52}$$

expresses F_I in terms of G_K's and lower order F_J's. Iterating, we can find the shuffle reciprocal. Now, since $\mathcal{S}^i$ satisfies the Leibnitz rule and $\mathcal{S}^i(1) = 0$,

$$0 = \mathcal{S}^i(G(\xi) \circ F(\xi)) = \mathcal{S}^i G \circ F + G \circ \mathcal{S}^i F \quad \Rightarrow \quad \mathcal{S}^i G = -G \circ \mathcal{S}^i F \circ G. \tag{53}$$

Thus the loop equations become linear equations for the graded shuffle reciprocal

$$\mathcal{S}^i F = -\xi^i. \tag{54}$$

This substantial simplification is absent for a generic matrix model whose Schwinger–Dyson operator is not a derivation of the graded shuffle product. This underscores the potential practical importance of the derivation property of the Schwinger–Dyson operator of

Yang–Mills matrix models, in a possible approximation method based on expanding concatenation around shuffle.

We can give another physical interpretation. The large-N limit is a 'classical' limit since $U(N)$ invariants stop fluctuating in this limit, though $\hbar = 1$ is held fixed. However, even this large-N 'classical' limit is difficult to solve, partly because we have non-commutative concatenation products left over. Replacing concatenation by commutative shuffle products may be thought of as taking a further classical limit. Indeed, when this is done, we found that the equations become linear! However, much work still needs to be done. In particular, for some models, we found that the loop equations are under-determined [12]. In those cases, the above equations have to be supplemented by non-anomalous Ward identities arising from Schwinger–Dyson equations that are naively $1/N^2$ suppressed [18]. We hope to report on more explicit calculations in a future publication.

Mathematically, together with [12], our work shows that it may be fruitful to think of the loop equations as living in the differential bi-algebra formed by concatenation, shuffle and their derivations. Roughly, the space of based oriented loops (modulo backtracking) on spacetime, Loop(M), is to be regarded as a free group on a continuously infinite number of generators labelled by the loops γ. The concatenation of loops and reversal of loops are the product and inverse operations in this gigantic free group. A typical function on Loop(M) is a Wilson loop expectation value with respect to a $U(N)$ connection. The space of functions on this Loop(M) is then automatically a commutative Hopf algebra under the commutative point-wise product of functions $(FG)(\gamma) = F(\gamma)G(\gamma)$ and comultiplication given by concatenation of loops: $(\Delta' G)(\gamma_1, \gamma_2) = G(\gamma_1\gamma_2)$. Formally, we also have a dual co-commutative Hopf algebra defined via the group algebra of the free group of loops. The Makeenko–Migdal loop equations of large-N Yang–Mills theory are defined in this space, using the path derivative of the area derivative (the analogue of our Schwinger–Dyson operator), which is a derivation of the point-wise product.

Hermitian multi-matrix models, regarded as discrete toy-models for gauge theory in physics, also provide a finitely generated toy-model for the above mathematical theory on loop space. The shuffle product plays the role of the commutative product of functions on Loop(M). It is obtained by expanding the Wilson loop in iterated integrals of gluon correlations. The coproduct in the discrete model involves concatenation of tensors $\left(\Delta'(\xi^I) = \delta^I_{JK}\xi^J \otimes \xi^K\right)$ rather than loops. One difference (since we are dealing with Hermitian rather than unitary matrices), is that we do not have inverses for the generators ξ^i to play the role of reversal of loop orientation. As a consequence, the group algebra of the free group of loops is now replaced by the concatenation algebra of tensors (free associative algebra), which is the monoid algebra of the free monoid. The free associative algebra could also have been obtained as the universal envelope of the free Lie algebra. Moreover, the left annihilation and its iterated commutators (including the physically relevant Schwinger–Dyson operator) form a free Lie algebra of derivations of the shuffle algebra. This suggests we should think of the Schwinger–Dyson operator of Yang–Mills theory as a vector field on the free group of loops on spacetime. Thus, despite being a discrete model, our setup preserves many of the algebraic and differential structures of the original theory on Loop(M), which is difficult to study directly. We hope to investigate this correspondence in greater detail.

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